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Hamiltonian relativistic two-body problem: center of mass and orbit reconstruction

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Abstract

After a short review of the history and problems of relativistic Hamiltonian mechanics with action-at-a-distance inter-particle potentials, we study isolated two-body systems in the rest-frame instant form of dynamics. We give explicit expressions of the relevant relativistic notions of center of mass, we determine the generators of the Poincare' group in presence of interactions and we show how to do the reconstruction of particles' orbits from the relative motion and the canonical non-covariant center of mass. In the case of a simple Coulomb-like potential model, it is possible to integrate explicitly the relative motion and show the two dynamical trajectories.

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1. Introduction

In Newtonian mechanics, the two-body problem is completely understood both in configuration and in phase space. The notions of absolute time and absolute space allow us to describe the two particles of mass m_i, $i = 1, 2$, with Euclidean position 3-vectors $\vec{x}_i$ and momenta $\vec{p}_i$ in an inertial frame. For an isolated two-body system the Hamiltonian $H = \sum_{i=1}^{2} \frac{\vec{p}_i^2}{2m_i} + V(|\vec{x}_1 - \vec{x}_2|)$ is the energy generator of the kinematical Galilei group, whose other generators are all interaction independent. With the point (both in coordinate and in momenta) canonical transformation $\vec{x} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1+m_2}$, $\vec{p} = \vec{p}_1 + \vec{p}_2$, $\vec{r} = \vec{x}_1 - \vec{x}_2$, $\vec{q} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$ we can separate the decoupled center of mass from the relative motion: the new Hamiltonian is $H = H_{\text{com}} + H_{\text{rel}}$ with $H_{\text{com}} = \frac{\vec{p}^2}{2m}$ $(m = m_1 + m_2)$ and $H_{\text{rel}} = \frac{\vec{q}^2}{2\mu} + V(|\vec{r}|)$ $(\mu = \frac{m_1 m_2}{m})$. The relative Hamiltonian H_{rel} governs the relative motion and, when its Hamilton equations have been solved, the trajectories of the particles are obtained with the inverse canonical transformation $\vec{x}_1 = \vec{x} + \frac{m_2}{m_1+m_2}\vec{r}$, $\vec{x}_2 = \vec{x} - \frac{m_1}{m_1+m_2}\vec{r}$, $\vec{p}_1 = \frac{1}{2}\vec{p} + \vec{q}$, $\vec{p}_2 = \frac{1}{2}\vec{p} - \vec{q}$. As a consequence the non-relativistic theory of orbits, for either 2 or N particles, is well understood and developed (see, for instance, [1]).

By contrast, in special relativity, where only Minkowski spacetime is absolute, where there is no absolute notion of simultaneity and where inertial frames are connected by the transformations generated by the kinematical Poincare' group, the situation is extremely more complicated and till now there is no complete self-consistent theory of orbits even for the two-body case. This is due to the facts that

(i) the particles' locations and momenta are now 4-vectors x_1^μ, p_1^μ, x_2^μ, p_2^μ,;

(ii) the momenta are not independent, but must satisfy mass-shell conditions (since a relativistic particle is an irreducible representation of the Poincare' group with mass m_i and a value of the spin (only scalar particles will be studied in this paper));

(iii) a simultaneity convention (for instance, Einstein's one identifying inertial frames) for the synchronization of distant clocks has to be introduced, so that the time components x_i^o are not independent;

(iv) the inter-particle interaction potentials appear in the boosts as well as in the energy generator;

(v) the structure of the Poincare' group implies that there is no definition of a relativistic 4-center of mass sharing all the properties of the non-relativistic 3-center of mass.

Since a clarification of all these problems has recently been obtained [2], and since it is not well known the extent to which consistent relativistic action at a distance (a-a-a-distance) theories have been developed, we want to illustrate these developments by using a simple two-body system with a scalar action-at-a-distance (a-a-a-d) interaction, for which a closed Poincare' algebra can be found in the rest-frame instant form of dynamics, as an example. By using the relativistic generalization [2] of the above quoted non-relativistic canonical basis, we will show that the potential appearing in the energy Hamiltonian (as with H_{rel}) determines the relative motion, while the potentials appearing in the Lorentz boosts (which disappear in the non-relativistic limit), together with the notion of the canonical non-covariant 4-center of mass, contribute to the reconstruction of the actual orbits of the two particles.

As a consequence for the first time we have full control on the relativistic theory of orbits and we can start to reformulate at the relativistic level the properties of the Newtonian theory of orbits.

In section 2, we give a brief history of the problems that have arisen in past attempts to define Hamiltonian relativistic mechanics with emphasis on several developments that are particularly relevant to the two-body problem discussed here. These developments include the instant form of dynamics with its two (external and internal) realizations of the Poincare' algebra, the three intrinsic notions of relativistic collective center-of-mass-like variables in both the realizations and the relativistic extension of the non-relativistic canonical transformation implementing the separation of the center of mass from the relative variables and how this can be used in general to do the reconstruction of particles' orbits from the relative motion and the canonical non-covariant center of mass. In section 3 there is the study of a simple two-body model with a-a-a-distance interaction which correctly reproduces the Poincare' algebra including potential-dependent boosts and energy generators, while in section 4 there is the determination of its orbits with an explicit integration of its equations of motion. A final discussion on the relativistic theory of orbits with its avoidance of the no-interaction theorem is given in the concluding section 5. Finally, in appendix A there is a review of the two-body models with first class constraints. A more detailed and historical discussion is given in a longer version of this paper in the archives [3].

2. Brief history of Hamiltonian relativistic mechanics and recent developments

Relativistic classical particle mechanics with a-a-a-d interactions and its Hamiltonian counterpart arose as an approximation to interactions with a finite time delay (like the electromagnetic one) and have been quite useful in the treatment of relativistic bound states with an instantaneous approximation of the kernels of field-theoretic equations like the Bethe–Salpeter equation. The starting points for Hamiltonian relativistic particle mechanics were the instant, front and point forms of relativistic Hamiltonian dynamics proposed by Dirac [4] (see also [5, 6]). This approach was an attempt to find canonical realizations of the Poincare' algebra such that some of the generators, called Hamiltonians (the energy and the boosts in the instant form), are not the direct sum of the corresponding ones for free particles.

The main obstacle in the development of models was the Currie–Jordan–Sudarshan no-interaction theorem [7], whose implication was the impossibility in theories with interactions for the canonical particle 4-positions to be 4-vectors, when their time components are put equal to the time of the reference inertial frame.

From numerous studies [8–14] it has become clear that relativistic particle mechanics has to be formulated by using Dirac's theory of constraints [15][1]: there must be as many mass-shell first class constraints (containing the potentials of the mutual interactions among the particles) as particles. The first consistent two-body model with two first class constraints depending upon a suitable potential was found by Droz-Vincent [16], Todorov [17] and Komar [18] simultaneously and independently (see appendix A; for $N \geqslant 3$ a closed form of the N first class constraints is not known). These studies led to the following problems, directly relevant to this paper.

(a) The study of two (more generally N)-particle configurations with a one-to-one correlation among the worldlines. This can be done by adding gauge fixing constraints, so that only the combination of the original constraints describing the mass spectrum of the global system of particles remains first class (moreover, there are $N-1$ pairs of second class constraints). Kalb and Van Alstine [19] and the authors of [20] developed consistent two-body models with second class constraints. The avoidance of the relative times in these models has been recently re-interpreted in [21] as the problem of the synchronization of the clocks associated to the individual particles.

(b) The identification of canonical bases containing a relativistic 4-center of mass and relativistic relative variables starting from the original canonical 4-vectors x_1^μ, p_1^μ, x_2^μ, p_2^μ. This was a highly non-trivial problem due to the lack of a unique notion of relativistic center of mass. If we use only the Poincare' generators of the N-particle system, it is possible to define *only three* such notions: a canonical non-covariant Newton–Wigner-like 3-center of mass [8, 11, 12], a non-canonical non-covariant Møller 3-center of energy [9] and a non-canonical covariant Fokker–Pryce 3-center of inertia [10, 11]. Each of these then has to be extended to 4-centers ($\tilde{x}^\mu$, R^μ, Y^μ, respectively). As shown in [2] a full understanding of these topics has been obtained in the framework of the *Wigner-covariant rest-frame instant form of dynamics*, developed in [22, 23] and explained in the following subsection. This instant form is a special case of *parametrized Minkowski theories* [20, 22], an approach developed to give a formulation of the N-body problem on arbitrary simultaneity 3-surfaces (corresponding to a convention for the synchronization of distant clocks [19]). In this way both the unknown closed form of first class constraints and the special choices of gauge fixings leading to second class ones are avoided. Moreover, the

[1] The symbol ≈ 0 (weakly equal to zero) means that the constraint has been used to get the equality. Let us remember that the constraints can be imposed only after the Poisson brackets are evaluated.

change of clock synchronization convention may be formulated as a *gauge transformation* not altering the physics, and there is no problem in introducing the electro-magnetic field when the particles are charged. The rest-frame instant form corresponds to the gauge choice of the 3+1 splitting whose simultaneity 3-surfaces are the intrinsic rest frame of the given configuration of the isolated system. For more details, see appendix B of [3].

(c) The identification of special models suited to the relativistic bound state problem. The model building [14, 22, 24–26] initially concentrated on the potentials in the energy Hamiltonian, which governs the relative motion (the canonical non-covariant 4-center of mass has free motion). The much harder problem to find the suitable potentials in the Lorentz boosts [6], so that the global Poincare' algebra is satisfied, was finally solved in [27] for charged scalar particles interacting with a dynamical electro-magnetic field (with Grassmann-valued electric charges to regularize the self-energies): in the sector of configurations without an independent radiation field, the Darwin potential to all order of $1/c^2$ appeared in the energy Hamiltonian and suitable related potentials in the boost Hamiltonians. In [28] analogous results (involving the Salpeter potential) were obtained for charged spinning particles (with Grassmann-valued spins implying Dirac spin-1/2 fermions after quantization). There are other approaches, not coming down from quantum field theory through instantaneous approximations to the Bethe–Salpeter equation, that arrive at Darwin Hamiltonians through $1/c^2$ and $1/c^4$ orders [29–32]: they do not use Grassmann-valued electric charges, but have a dependence on higher accelerations and need some regularization of the self-energies.

2.1. The inertial rest-frame instant form of dynamics

In the rest-frame instant form of dynamics, Minkowski spacetime[2] is foliated with inertial hyper-planes (simultaneity 3-surfaces called *Wigner hyper-planes*) orthogonal to the constant 4-velocity $u^\mu = P^\mu/M, u^2 = 1$, of a special inertial observer; it can be shown [21, 22] that the 4-momentum P^μ, canonically conjugate to the canonical non-covariant 4-center of mass $\tilde{x}^\mu$, is weakly equal to the conserved 4-momentum P^μ_{sys} of the 2-particle system ($M = \sqrt{P^2_{\text{sys}}}$ is the invariant mass). The observer-dependent 4-coordinates $(\tau; \vec{\sigma})$ are the proper time τ of this observer and 3-coordinates on the Wigner hyper-planes having the observer as origin $\vec{\sigma} = 0$ for every τ.

Therefore, the Wigner hyperplane Σ_τ at time τ is the intrinsic rest frame of the isolated system at time τ. With respect to an arbitrary inertial frame, the Wigner hyper-planes are described by the following embedding:

$$z^\mu(\tau, \vec{\sigma}) = x^\mu_s(\tau) + \epsilon^\mu_r(u(P))\sigma^r, \tag{1}$$

with $x^\mu_s(\tau)$ being the worldline of our arbitrary inertial observer. The space-like 4-vectors $\epsilon^\mu_r(u(P))$ together with the time-like one $\epsilon^\mu_o(u(P))$ are the columns of the standard Wigner boost for time-like Poincare' orbits that sends the time-like 4-vector P^μ to its rest-frame form $\mathring{P}^\mu = \sqrt{P^2}(1; \vec{0})$:

$$\epsilon^\mu_o(u(P)) = u^\mu(P) = P^\mu/\sqrt{P^2}, \qquad \epsilon^\mu_r(u(P)) = \left(-u_r(P); \delta^i_r - \frac{u^i(P)u_r(P)}{1 + u^o(P)}\right). \tag{2}$$

Since we are in the rest frame, we have $\tau \equiv T_s = u(P) \cdot \tilde{x} = u(P) \cdot x_s$ as the scalar rest time of the inertial observer, whose worldline is given by

$$x^\mu_s(\tau) = x^\mu(0) + u^\mu(P)\tau. \tag{3}$$

[2] We use the metric $\eta_{\mu\nu} = (+ - - -)$.

In the rest-frame instant form the particles' 4-coordinates, describing their worldlines, and the associated momenta are

$$x_i^\mu(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau)) = x_s^\mu(\tau) + \epsilon_r^\mu(u(P))\eta_i^r(\tau),$$
$$p_i^\mu(\tau) = \sqrt{m_i^2 + \vec{\kappa}_i^2(\tau)}u^\mu(P) + \epsilon_r^\mu(u(P))\kappa_{ir}(\tau) \Rightarrow p_i^2 = m_i^2, \qquad i = 1, 2. \tag{4}$$

The momenta $p_i^\mu(\tau)$ are suitable solutions of the mass-shell constraints; in this approach, all the first class constraints have been solved explicitly to determine the single particle energies. As a consequence, each particle must have a definite sign of the energy (we only consider positive energies). The total system momentum is

$$P_{\text{sys}}^\mu = p_1^\mu(\tau) + p_2^\mu(\tau) = P^\mu + \epsilon_r^\mu(u(P))(\kappa_{1r}(\tau) + \kappa_{2r}(\tau)) \approx P^\mu,$$
$$P^\mu = \big(\sqrt{m_1^2 + \vec{\kappa}_1^2(\tau)} + \sqrt{m_2^2 + \vec{\kappa}_2^2(\tau)}\big)u^\mu(P). \tag{5}$$

As we see from this and equation (2), P^μ coincides with the conserved 4-momentum P_{sys}^μ of the 2-particle system in its rest frame, defined by the three first class constraints $\vec{p} = \sum_i \vec{\kappa}_i \approx 0$.

This shows that inside the Wigner hyper-planes, the two particles are described by the 12 Wigner spin-1 3-vectors $\vec{\eta}_i(\tau), \vec{\kappa}_i(\tau)$ as independent canonical variables $\big[\big\{\eta_i^r(\tau), \kappa_{js}(\tau)\big\} = \delta_{ij}\delta_s^r, \big\{\eta_i^r(\tau), \eta_j^s(\tau)\big\} = \{\kappa_{ir}(\tau), \kappa_{js}(\tau)\} = 0\big]$. To them we must add $P^\mu = p_1^\mu + p_2^\mu$ and a canonically conjugate collective variable $\tilde{x}^\mu$, the *external* canonical non-covariant 4-center of mass ($M = \sqrt{P^2}$)[2] given by

$$\begin{aligned}
\tilde{x}^\mu(\tau) &= (\tilde{x}^o(\tau); \vec{\tilde{x}}(\tau)) = z^\mu(\tau, \vec{\tilde{\sigma}}) \\
&= x_s^\mu(\tau) - \frac{1}{M(P^o + M)}\left[P_\nu S^{\nu\mu} + M\left(S^{o\mu} - S^{o\nu}\frac{P_\nu P^\mu}{M^2}\right)\right], \\
\{\tilde{x}^\mu, P^\nu\} &= -\eta^{\mu\nu}, \\
S^{\mu\nu} &= \big[u^\mu(P)\epsilon_r^\nu(u(P)) - u^\nu(P)\epsilon_r^\mu(u(P))\big]\bar{S}^{or} + \epsilon_r^\mu(u(P))\epsilon_s^\nu(u(P))\bar{S}^{rs}, \\
\bar{S}^{rs} &\equiv \big(\eta_1^r\kappa_1^s - \eta_1^s\kappa_1^r + \eta_2^r\kappa_2^s - \eta_2^s\kappa_2^r\big) = \varepsilon^{rst}j_t, \\
\bar{S}^{or} &\equiv -\eta_1^r\sqrt{m_1^2 + \vec{\kappa}_1^2} - \eta_2^r\sqrt{m_2^2 + \vec{\kappa}_2^2} = k^r.
\end{aligned} \tag{6}$$

The point with coordinates $\tilde{x}^\mu(\tau)$ is the decoupled canonical *external 4-center of mass* (at the non-relativistic level, it is the free center of mass $\vec{x}$ with Hamiltonian $H_{\text{com}} = \frac{\vec{p}^2}{2m}$), playing the role of a kinematical external 4-center of mass and of a decoupled observer with his parametrized clock (*point particle clock*). It describes the decoupled collective degrees of freedom of the isolated system.

An important feature of the rest-frame instant form is that it separates the relativistic center of mass from the relative motion by means of a *splitting* of the description of the isolated system into an *external* one and an *internal* one.

(a) The *external* description of the isolated system as a point particle clock is concerned with the embedding of the Wigner hyper-planes into Minkowski spacetime from the point of view of our generic inertial observer. There is an *external* realization of the Poincare' algebra, which governs the covariance properties of Wigner hyper-planes under Poincare' transformations (Λ, a). It can be shown [2] that its generators have the following form

[$M=\sqrt{P^2}$, while $l, m\ldots$ are Euclidean indices; $r, s\ldots$ are Wigner spin-1 indices; $\tilde{S}^{\mu\nu}$ is given in equation (6)]

$$\begin{aligned}
&P^\mu, \qquad J^{\mu\nu} = x_s^\mu P^\nu - x_s^\nu P^\mu + S^{\mu\nu} = \tilde{x}^\mu P^\nu - \tilde{x}^\nu P^\mu + \tilde{S}^{\mu\nu},\\
&P^o = \sqrt{M^2+\vec{P}^2},\\
&J^{lm} = \tilde{x}^l P^m - \tilde{x}^m P^l + \delta^{lr}\delta^{ms}\epsilon^{rsu}\bar{S}^u,\\
&K^l = J^{ol} = \tilde{x}^o P^l - \tilde{x}^l\sqrt{M^2+\vec{P}^2} - \frac{\delta^{lr}P^s\epsilon^{rsu}\bar{S}^u}{M+\sqrt{M^2+\vec{P}^2}},\\
&\{P^\mu, P^\nu\} = 0, \qquad \{P^\mu, J^{\alpha\beta}\} = \eta^{\mu\alpha}P^\beta - \eta^{\mu\beta}J^\alpha,\\
&\{J^{\mu\nu}, J^{\alpha\beta}\} = \eta^{\mu\alpha}J^{\nu\beta} - \eta^{\mu\beta}J^{\nu\alpha} - \eta^{\nu\alpha}J^{\mu\beta} + \eta^{\nu\beta}J^{\mu\alpha}.
\end{aligned} \tag{7}$$

Note that both $\tilde{L}^{\mu\nu} = \tilde{x}^\mu P^\nu - \tilde{x}^\nu P^\mu$ and $\tilde{S}^{\mu\nu}$ are conserved.

Let us remark that this realization is universal in the sense that it depends on the nature of the isolated system only through the invariant mass M (which in turn depends on the relative variables and on the type of interaction).

(b) The *internal* description concerns the relative degrees of freedom of the isolated system inside the Wigner hyper-plane (replacing the absolute Newtonian Euclidean 3-space containing the isolated system). In order not to have a double counting of the center-of-mass degrees of freedom there is the *rest-frame condition*, which implies the existence of the three first class constraints on the internal 3-momentum $\vec{p} = \vec{\kappa}_1 + \vec{\kappa}_2 \approx 0$.[3] This implies that a collective 3-variable (the *internal* 3-center of mass) inside each Wigner hyper-plane can be eliminated, so that only six internal relative canonical variables are independent. Since the spin tensor $\bar{S}^{AB}$ satisfies a Lorentz algebra we can build an *unfaithful internal* realization (in the sense that some of the generators weakly vanish) of the Poincaré algebra, acting inside the Wigner hyperplane, whose generators are [2]

$$\begin{aligned}
&M = p^\tau = \sqrt{m_1^2+\vec{\kappa}_1^2} + \sqrt{m_2^2+\vec{\kappa}_2^2}, &&\vec{p} = \vec{\kappa}_1 + \vec{\kappa}_2(\approx 0),\\
&\vec{j} = \vec{\eta}_1 \times \vec{\kappa}_2, &&j^r = \bar{S}^r = \frac{1}{2}\epsilon^{ruv}\bar{S}^{uv},\\
&\vec{k} = -\sqrt{m_1^2+\vec{\kappa}_1^2}\,\vec{\eta}_1 - \sqrt{m_2^2+\vec{\kappa}_2^2}\,\vec{\eta}_2, &&k^r = j^{or} = \bar{S}^{or}.
\end{aligned} \tag{8}$$

They satisfy the Poincare' algebra (like the external ones)

$$\begin{aligned}
&[p^\tau, \mathbf{p}]=0=[p_l, p_m], &&[p_l, k_m] = \delta_{lm}p^\tau, &&[p^\tau, \mathbf{k}]=[p^\tau, \mathbf{j}]=0,\\
&[j_l, j_m] = \varepsilon_{lmn}j_n, &&[j_l, k_m] = \varepsilon_{lmn}k_n, &&[k_l, k_m] = -\varepsilon_{lmn}j_n.
\end{aligned} \tag{9}$$

The Poisson brackets $[p_l, k_m] = \delta_{lm}p^\tau$ show clearly that for the interacting case, the presence of interaction potentials in the invariant mass $p^\tau = M$ requires the presence of potentials in the boost generators k_i.

As shown in [2, 22], the natural gauge fixings to be added to the rest-frame constraints $\vec{p} \approx 0$ are the vanishing of the internal boosts $\vec{k} \approx 0$. It implies the uniqueness of the collective 4-velocity $\dot{x}_s^\mu(\tau) = \dot{\tilde{x}}^\mu(\tau) = u^\mu(P)$, so that there is no *classical zitterbewegung* in the associated worldlines. Moreover, it can be shown that the inertial observer $x_s^\mu(\tau)$ can be identified with the covariant Fokker–Pryce center of inertia Y^μ.

As a consequence of equations (4) and (6), in the rest-frame instant form the standard 16 variables $x_1^\mu, p_1^\mu, x_2^\mu, p_2^\mu$ are re-expressed in terms of the 8 variables x_s^μ (or $\tilde{x}^\mu$), $P^\mu = Mu^\mu(P)$ and the 12 variables $\vec{\eta}_1, \vec{\kappa}_1, \vec{\eta}_2, \vec{\kappa}_2$ restricted by the rest-frame condition $\vec{p} = \vec{\kappa}_1 + \vec{\kappa}_2 \approx 0$ and

[3] In the non-relativistic limit, we get the standard description with $\vec{\eta}_i = \vec{x}_i, \vec{\kappa}_i = \vec{p}_i$ in the Newtonian rest frame $\vec{p} \approx 0$.

gauge condition $\vec{k} \approx 0$. (Note that $\vec{p}$ is distinct from the spatial component of $\vec{P}$.) Therefore we have 8+12− 6 = 14 variables, the lacking 2 variables to arrive at 16 are the relative time (which vanishes due to the clock synchronization) and relative energy (vanishing because, as shown in equations (4), the particles are on mass shell).

In [27] we found the generators for a system of N-interacting Grassmann charged particles and electro-magnetic fields. The internal Hamiltonian and boosts have the following form for $N = 2$ $[c(\vec{\sigma}) = -1/4\pi|\vec{\sigma}|]$

$$\begin{aligned} M &= \sqrt{m_1^2 + (\vec{\kappa}_1(\tau) - Q_1\vec{A}_\perp(\tau, \vec{\eta}_1(\tau)))^2} + \sqrt{m_2^2 + (\vec{\kappa}_2(\tau) - Q_2\vec{A}_\perp(\tau, \vec{\eta}_2(\tau)))^2} \\ &\quad + \frac{Q_1 Q_2}{4\pi \mid \vec{\eta}_1(\tau) - \vec{\eta}_2(\tau) \mid} + \int d^3\sigma \frac{1}{2}\big[\vec{E}_\perp^2 + \vec{B}^2\big](\tau, \vec{\sigma}), \\ k^r &= -\sum_{i=1}^{2} \eta_i^r(\tau)\sqrt{m_i^2 + (\vec{\kappa}_i(\tau) - Q_i\vec{A}_\perp(\tau, \vec{\eta}_i(\tau)))^2} \\ &\quad + \sum_{i=1}^{2}\left[Q_1 Q_2 \sum_{i\neq j}\left(\frac{1}{\nabla_{\vec{\eta}_j}}\frac{\partial}{\partial \eta_j^r} c(\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)) - \eta_j^r(\tau) c(\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau))\right)\right. \\ &\quad \left. + Q_i \int d^3\sigma E_\perp^r(\tau, \vec{\sigma}) c(\vec{\sigma} - \vec{\eta}_i(\tau))\right] - \frac{1}{2}\int d^3\sigma \sigma^r \big[\vec{E}_\perp^2 + \vec{B}^2\big](\tau, \vec{\sigma}), \end{aligned} \tag{10}$$

In the sector without independent radiation field it can be shown [27] that the Coulomb potential is replaced by the classical Darwin potential. Let us remark that starting from classical electrodynamics we arrive at a Coulomb potential additive to the square roots, and not living inside them like in the toy model at the end of appendix A, whose rest-frame instant form will be studied in section 4.

2.2. *The problem of the relativistic center of mass*

As shown in [2], given an isolated system with an associated realization of the Poincare' algebra, only three notions of collective 3-variables (coinciding only in the rest frame) can be built in term of them (namely without introducing external variables). This is done by using the group theoretical methods of [6]. They are (i) a canonical non-covariant center of mass (or *center of spin*); it is the classical analogue [11, 12] of the Newton–Wigner position operator [8], (ii) a non-canonical non-covariant Møller *center of energy* [9] (it is the non-relativistic prescription with the particle energies replacing their masses) and (iii) a non-canonical covariant Fokker–Pryce *center of inertia* [10, 11], leading to a 4-vector defining a frame-independent worldline.

However, none of these candidates for the relativistic center of mass has all the properties of the non-relativistic center of mass. Since in the rest-frame instant form of dynamics we have both an *internal* and an *external* realization of the Poincare' algebra, there will be three internal collective 3-variables (they are gauge variables to be eliminated by $\vec{k} \approx 0$) and three external collective 3-variables (to be extended to suitable collective 4-variables).

The canonical (Wigner spin-1) *internal* 3-center of mass (or *center of spin*) $\vec{q}_+$ is

$$\begin{aligned} \vec{q}_+ &= -\frac{\vec{k}}{\sqrt{M^2 - \vec{p}^2}} + \frac{\vec{j} \times \vec{p}}{\sqrt{M^2 - \vec{p}^2}(M + \sqrt{M^2 - \vec{p}^2})} \\ &\quad + \frac{\vec{k}\cdot\vec{p}\,\vec{p}}{M\sqrt{M^2 - \vec{p}^2}\left(M + \sqrt{M^2 - \vec{p}^2}\right)} \approx -\frac{\vec{k}}{\sqrt{M^2 - \vec{p}^2}}, \end{aligned}$$

$$
\begin{aligned}
&M = \sqrt{m_1^2 + \vec{\kappa}_1^2} + \sqrt{m_2^2 + \vec{\kappa}_2^2},\\
&\vec{k} = -\sqrt{m_1^2 + \vec{\kappa}_1^2}\,\vec{\eta}_1 - \sqrt{m_2^2 + \vec{\kappa}_2^2}\,\vec{\eta}_2,\\
&\{\vec{q}_+, M\} = \frac{\vec{p}}{M}, \qquad \{q_+^r, q_+^s\} = 0, \qquad \{q_+^r, p^s\} = \delta^{rs},
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&\vec{S}_q = \vec{j} - \vec{q}_+ \times \vec{p} = \frac{M\vec{j}}{\sqrt{M^2 - \vec{p}^2}} + \frac{\vec{k} \times \vec{p}}{\sqrt{M^2 - \vec{p}^2}} - \frac{\vec{j} \cdot \vec{p}\,\vec{p}}{\sqrt{M^2 - \vec{p}^2}\,(M + \sqrt{M^2 - \vec{p}^2})} \approx \vec{\bar{S}} = \vec{j},\\
&\{\vec{S}_q, \vec{p}\} = \{\vec{S}_q, \vec{q}_+\} = 0, \qquad \{S_q^r, S_q^s\} = \epsilon^{rsu} S_q^u.
\end{aligned}
\tag{12}
$$

Note that in the non-relativistic limit, $\vec{q}_+$ tends weakly to the non-relativistic center of mass $\vec{q}_{nr} = \frac{m_1\vec{\eta}_i + m_2\vec{\eta}_2}{m_1+m_2}$. Moreover, the rest-frame conditions $\vec{p} \approx 0$ imply that the internal Møller 3-center of energy and the internal Fokker–Pryce 3-center of inertia weakly coincide with $\vec{q}_+$. The natural gauge fixings $\vec{k} \approx 0$ imply $\vec{q}_+ \approx 0$, so that the only non-zero generators of the internal Poincare' algebra are M and $\vec{j} = \vec{\bar{S}}$ containing all the information about the isolated system and generate the dynamical U(2) algebra of [33]. As is evident from equations (7), the external Poincare' generators also depend only on the generators of this U(2) algebra.

On the other hand, from the external realization of the Poincare' algebra we get the following *external* canonical center-of-mass 3-variable:

$$
\vec{q}_s = \vec{\tilde{x}} - \frac{\vec{P}}{P^o}\tilde{x}^o, \qquad \{q_s^r, q_s^s\} = 0.
\tag{13}
$$

As shown in [2], the requirement that the relations $\tau \equiv T_s = u(P) \cdot x_s$ holds on Wigner hyper-planes allows us to extend the external collective 3-variable $\vec{q}_s$ to the *external* canonical non-covariant 4-center of mass $\tilde{x}^\mu$ (a frame-dependent pseudo-worldline, whose intersection with the Wigner hyper-plane has 3-coordinate $\tilde{\sigma}^r$) defined as

$$
\tilde{x}^\mu = (\tilde{x}^o; \vec{\tilde{x}}) = \left(\tilde{x}^o; \vec{q}_s + \frac{\vec{p}}{p^o}\tilde{x}^o\right) \equiv x_s^\mu + \epsilon_u^\mu(u(P))\tilde{\sigma}^u.
\tag{14}
$$

See [2, 3] for the definition of the external Møller center of energy and Fokker–Pryce center of inertia. The three external collective 4-variables have the same 4-velocity and coincide in the Lorentz rest frame where $\mathring{P}^\mu = M(1; \vec{0})$

Since we are in an instant form of dynamics, in the presence of interactions among the constituents of the isolated system only the internal generators M and $\vec{k}$ will contain the interaction potentials, $M \mapsto M_{(\text{int})}, \vec{k} \mapsto \vec{k}_{(\text{int})}$, but only those inside $M_{(\text{int})}$ contribute to the external Poincare' (and U(2)) algebra. Instead, as shown in section 4, the potentials inside $\vec{k}_{(\text{int})}$ contribute to the elimination of the internal 3-centers by means of the gauge fixings $\vec{k}_{(\text{int})} \approx 0$.

2.3. The canonical transformation to relative variables

From the previous discussion it is clear that in the rest-frame instant form the 2-body problem in the free case is described by the 20 canonical variables $\tilde{x}^\mu, P^\mu, \vec{\eta}_1, \vec{\kappa}_1, \vec{\eta}_2, \vec{\kappa}_2$, restricted by the six conditions $\vec{p} \approx 0, \vec{q}_+ \approx 0$ and with $P^\mu = Mu^\mu(P)$, $M = \sqrt{m_1^2 + \vec{\kappa}_1^2} + \sqrt{m_2^2 + \vec{\kappa}_2^2}$ and $\tau \equiv T_s = u(P) \cdot \tilde{x}$. There are only 12 independent canonical variables like in the non-relativistic case.

We now have to find the canonical transformation

$$\begin{array}{|c|c|}\hline \tilde{x}^\mu & \vec{\eta}_i \\ \hline P^\mu & \vec{\kappa}_i \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|c|}\hline \tilde{x}^\mu & \vec{q}_+ & \vec{\rho}_q \\ \hline P^\mu & \vec{p} & \vec{\pi}_q \\ \hline \end{array}\ , \qquad \vec{p} \approx 0, \qquad \vec{q}_+ \approx 0, \tag{15}$$

defining the six relativistic relative variables $\vec{\rho}_q, \vec{\pi}_q$, so that the spin (barycentric angular momentum) becomes $\vec{S}_q = \vec{\rho}_q \times \vec{\pi}_q$. Let us stress that this cannot be a point transformation, because of the momentum dependence of the relativistic internal center of mass $\vec{q}_+$.

Since $\vec{q}_+$ and $\vec{p}$ are known from equations (11) and (8) respectively, we have only to find the internal conjugate variables appearing in the canonical transformation (15). They have been determined in [2] by using the technique (the Gartenhaus–Schwarz transformation) of [34] and starting from a set of canonical variables defined in [22]. In terms of the naive *internal* center-of-mass variable $\vec{\eta}_+ = \frac{1}{2}(\vec{\eta}_1+\vec{\eta}_2)$, we defined relative variables $\vec{\rho}, \vec{\pi}$ based on the following family of point canonical transformations:

$$\begin{array}{|c|}\hline \vec{\eta}_i \\ \hline \vec{\kappa}_i \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|}\hline \vec{\eta}_+ & \vec{\rho} \\ \hline \vec{p} & \vec{\pi} \\ \hline \end{array}\ ,$$

$$\begin{aligned} &\vec{\eta}_1 = \vec{\eta}_+ + \tfrac{1}{2}\vec{\rho}, \qquad \vec{\eta}_2 = \vec{\eta}_+ - \tfrac{1}{2}\vec{\rho}, \qquad \vec{\kappa}_1 = \tfrac{1}{2}\vec{p} + \vec{\pi}, \qquad \vec{\kappa}_2 = \tfrac{1}{2}\vec{p} - \vec{\pi}, \\ &\vec{\eta}_+ = \tfrac{1}{2}(\vec{\eta}_1+\vec{\eta}_2), \qquad \vec{p} = \vec{\kappa}_1 + \vec{\kappa}_2 \approx 0, \qquad \vec{\rho} = \vec{\eta}_1 - \vec{\eta}_2, \qquad \vec{\pi} = \tfrac{1}{2}(\vec{\kappa}_1 - \vec{\kappa}_2), \\ &\left\{\eta_i^r, \kappa_j^s\right\} = \delta_{ij}\delta^{rs}, \qquad \left\{\eta_+^r, p^s\right\} = \delta^{rs}, \qquad \{\rho^r, \pi^s\} = \delta^{rs}. \end{aligned} \tag{16}$$

The closed form of the canonical transformation (15) for arbitrary N was given in terms of ρ and π in appendix B of [35]. The transformation is *point in the momenta* but, unlike the non-relativistic case, *non-point* in the configurational variables. Explicitly, we have for $N = 2$

$$\begin{aligned} M &= \sqrt{m_1^2 + \vec{\kappa}_1^2} + \sqrt{m_2^2 + \vec{\kappa}_2^2}, \qquad \vec{S}_q = \vec{\rho}_q \times \vec{\pi}_q, \\ \vec{q}_+ &= \frac{\sqrt{m_1^2 + \vec{\kappa}_1^2}\vec{\eta}_1 + \sqrt{m_2^2 + \vec{\kappa}_2^2}\vec{\eta}_2}{\sqrt{M^2 - \vec{p}^2}} + \frac{(\vec{\eta}_1 \times \vec{\kappa}_1 + \vec{\eta}_2 \times \vec{\kappa}_2) \times \vec{p}}{\sqrt{M^2 - \vec{p}^2}(M + \sqrt{M^2 - \vec{p}^2})} \\ &\quad - \frac{\left(\sqrt{m_1^2 + \vec{\kappa}_1^2}\vec{\eta}_1 + \sqrt{m_2^2 + \vec{\kappa}_2^2}\vec{\eta}_2\right) \cdot \vec{p}\,\vec{p}}{M\sqrt{M^2 - \vec{p}^2}\left(M + \sqrt{M^2 - \vec{p}^2}\right)}, \\ \vec{p} &= \vec{\kappa}_1 + \vec{\kappa}_2 \approx 0, \\ \vec{\pi}_q &= \vec{\pi} - \frac{\vec{p}}{\sqrt{M^2 - \vec{p}^2}}\left[\frac{1}{2}\left(\sqrt{m_1^2 + \vec{\kappa}_1^2} - \sqrt{m_2^2 + \vec{\kappa}_2^2}\right)\right. \\ &\quad \left. - \frac{\vec{p} \cdot \vec{\pi}}{\vec{p}^2}\left(M - \sqrt{M^2 - \vec{p}^2}\right)\right] \approx \vec{\pi} = \frac{1}{2}(\vec{\kappa}_1 - \vec{\kappa}_2), \\ \vec{\rho}_q &= \vec{\rho} + \left(\frac{\sqrt{m_1^2 + \vec{\kappa}_1^2}}{\sqrt{m_2^2 + \vec{\pi}_q^2}} + \frac{\sqrt{m_2^2 + \vec{\kappa}_2^2}}{\sqrt{m_1^2 + \vec{\pi}_q^2}}\right)\frac{\vec{p} \cdot \vec{\rho}\vec{\pi}_q}{M\sqrt{M^2 - \vec{p}^2}} \approx \vec{\rho} = \vec{\eta}_1 - \vec{\eta}_2, \\ &\Rightarrow M = \sqrt{\mathcal{M}^2 + \vec{p}^2} \approx \mathcal{M} = \sqrt{m_1^2 + \vec{\pi}_q^2} + \sqrt{m_2^2 + \vec{\pi}_q^2}, \\ \vec{q}_+ &\approx \frac{\vec{\eta}_1\sqrt{m_1^2 + \vec{\pi}^2} + \vec{\eta}_2\sqrt{m_2^2 + \vec{\pi}^2}}{\mathcal{M}}. \end{aligned} \tag{17}$$

The inverse canonical transformation is ($i = 1, 2$)

$$\vec{\eta}_i = \vec{q}_+ - \frac{\vec{S}_q \times \vec{p}}{\sqrt{\mathcal{M}^2 + \vec{p}^2}\left(\mathcal{M} + \sqrt{\mathcal{M}^2 + \vec{p}^2}\right)} + \frac{1}{2}\left[(-1)^{i+1} - \frac{2\mathcal{M}\vec{\pi}_q \cdot \vec{p} + \left(m_1^2 - m_2^2\right)\sqrt{\mathcal{M}^2 + \vec{p}^2}}{\mathcal{M}^2\sqrt{\mathcal{M}^2 + \vec{p}^2}}\right],$$

$$\left[\vec{\rho}_q - \frac{\vec{\rho}_q \cdot \vec{p}\vec{\pi}_q}{\mathcal{M}\sqrt{\mathcal{M}^2 + \vec{p}^2}\left(\frac{\sqrt{m_1^2+\vec{\kappa}_1^2}}{\sqrt{m_2^2+\vec{\pi}_q^2}} + \frac{\sqrt{m_2^2+\vec{\kappa}_2^2}}{\sqrt{m_1^2+\vec{\pi}_q^2}}\right)^{-1} + \vec{\pi}_q \cdot \vec{p}}\right]$$

$$\approx \vec{q}_+ + \frac{1}{2}\left[(-1)^{i+1} - \frac{m_1^2 - m_2^2}{\mathcal{M}^2}\right]\vec{\rho}_q \approx \frac{1}{2}\left[(-1)^{i+1} - \frac{m_1^2 - m_2^2}{\mathcal{M}^2}\right]\vec{\rho},$$

$$\vec{\kappa}_i = \left[\frac{1}{2} + \frac{(-1)^{i+1}}{\mathcal{M}\sqrt{\mathcal{M}^2 + \vec{p}^2}}\left(\vec{\pi}_q \cdot \vec{p}\left[1 - \frac{\mathcal{M}}{\vec{p}^2}\left(\sqrt{\mathcal{M}^2 + \vec{p}^2} - \mathcal{M}\right)\right]\right.\right.$$

$$\left.\left. + \left(m_1^2 - m_2^2\right)\sqrt{\mathcal{M}^2 + \vec{p}^2}\right)\right]\vec{p} + (-1)^{i+1}\vec{\pi}_q \approx (-1)^{i+1}\vec{\pi}_q \approx (-)^{i+1}\vec{\pi},$$

$$\Rightarrow \vec{\kappa}_i^2 \approx \vec{\pi}^2. \tag{18}$$

In equations (18), we used explicitly the gauge fixing $\vec{q}_+ \approx 0$.

As shown in [22, 36] and their bibliography, a-a-a-d interactions inside the Wigner hyperplane may be introduced either under (scalar and vector potentials) or outside (scalar potential like the Coulomb one) the square roots appearing in the free Hamiltonian. Since a Lagrangian density in the presence of action-at-a-distance mutual interactions is not known and since we are working in an instant form of dynamics, the potentials in the constraints restricted to hyper-planes must be introduced *by hand* (see, however, [27] for their evaluation starting from the Lagrangian density for the electro-magnetic interaction). The only restriction is that the Poisson brackets of the modified constraints' generators must generate the same algebra of the free ones.

In the rest-frame instant form, the most general two-body Hamiltonian with action-at-a-distance interactions is

$$M_{(\mathrm{int})} = \sqrt{m_1^2 + U_1 + [\vec{\kappa}_1 - \vec{V}_1]^2} + \sqrt{m_2^2 + U_2 + [\vec{\kappa}_2 - \vec{V}_2]^2} + V, \tag{19}$$

where $U_i = U_i(\vec{\kappa}_1, \vec{\kappa}_2, \vec{\eta}_1 - \vec{\eta}_2)$, $\vec{V}_i = \vec{V}_i(\vec{\kappa}_{j\neq i}, \vec{\eta}_1 - \vec{\eta}_2)$, $V = V_o(|\vec{\eta}_1 - \vec{\eta}_2|) + V'(\vec{\kappa}_1, \vec{\kappa}_2, \vec{\eta}_1 - \vec{\eta}_2)$.

If we use the canonical transformation (15) defining the relativistic canonical internal 3-center of mass (now $\vec{q}_+^{(\mathrm{int})}$ is interaction dependent) and relative variables on the Wigner hyperplane, together with the rest-frame conditions $\vec{p} \approx 0$, the rest-frame Hamiltonian for the relative motion becomes

$$M_{(\mathrm{int})} \approx \sqrt{m_1^2 + \tilde{U}_1 + [\vec{\pi}_q - \tilde{\vec{V}}_1]^2} + \sqrt{m_2^2 + \tilde{U}_2 + [-\vec{\pi}_q - \tilde{\vec{V}}_2]^2} + \tilde{V}, \tag{20}$$

where

$$\begin{aligned} \tilde{U}_i &= U_i(\vec{\pi}_q, \vec{\rho}_q), & \tilde{V} &= V_o(|\vec{\rho}_q|) + V'(\vec{\pi}_q, \vec{\rho}_q), \\ \tilde{\vec{V}}_1 &= \vec{V}_1(-\vec{\pi}_q, \vec{\rho}_q), & \tilde{\vec{V}}_2 &= \vec{V}_2(\vec{\pi}_q, \vec{\rho}_q). \end{aligned} \tag{21}$$

In order to build a realization of the internal Poincare' group, besides $M_{(\mathrm{int})}$ we need to know the potentials appearing in the internal boosts $\vec{k}_{(\mathrm{int})}$ (being an instant form, $\vec{p} \approx 0$ and $\vec{j}$ are interaction-independent generators). Since the 3-center $\vec{q}_+$ becomes interaction dependent, the final canonical basis $\vec{q}_+$, $\vec{p}$, $\vec{\rho}_q$, $\vec{\pi}_q$ is *not explicitly known in the interacting case*.

For an isolated system, however, we have $M = \sqrt{\mathcal{M}^2 + \vec{p}^2} \approx \mathcal{M}$ with $\mathcal{M}$ independent of $\vec{q}_+$ ($\{M, \vec{p}\} = 0$ in the internal Poincare' algebra). This suggests that the same result should hold true even in the interacting case. Indeed, by its definition, the Gartenhaus–Schwartz transformation [2, 34] gives $\vec{\rho}_q \approx \vec{\rho}$, $\vec{\pi}_q \approx \vec{\pi}$ also in presence of interactions, so that we get

$$\begin{aligned} M_{(\mathrm{int})}|_{\vec{p}=0} &= \Big(\sqrt{m_1^2 + U_1 + (\vec{\kappa}_1 - \vec{V}_1)^2} + \sqrt{m_2^2 + U_2 + (\vec{\kappa}_2 - \vec{V}_2)^2} + V\Big)\Big|_{\vec{p}=0} \\ &= \sqrt{\mathcal{M}^2_{(\mathrm{int})} + \vec{p}^2}|_{\vec{p}=0} = \mathcal{M}_{(\mathrm{int})}|_{\vec{p}=0} \\ &= \sqrt{m_1^2 + \tilde{U}_1 + (\vec{\kappa}_1 - \vec{\tilde{V}}_1)^2} + \sqrt{m_2^2 + \tilde{U}_2 + (\vec{\kappa}_2 - \vec{\tilde{V}}_2)^2} + \tilde{V}, \end{aligned} \tag{22}$$

where the potentials $\tilde{U}_i$, $\vec{\tilde{V}}_i$, $\tilde{V}$ are now functions of $\vec{\pi}_q^2$, $\vec{\pi}_q \cdot \vec{\rho}_q$, $\vec{\rho}_q^2$ respectively.

Unlike in the non-relativistic case, the canonical transformation (17) becomes *interaction dependent* (not even a point transformation in the momenta), since $\vec{q}_+$ is determined by a set of Poincare' generators depending on the interactions. The only thing to do in the generic situation is therefore to use the free relative variables (17) even in the interacting case. We cannot impose anymore, however, the natural gauge fixings $\vec{q}_+ \approx 0$ ($\vec{k} \approx 0$) of the free case, since it is replaced by $\vec{q}_+^{(\mathrm{int})} \approx 0$ (namely by $\vec{k}_{(\mathrm{int})} \approx 0$). Once written in terms of the canonical variables (17) of the free case, the equations $\vec{k}_{(\mathrm{int})} \approx 0$ *can be solved* for $\vec{q}_+$, which takes a form $\vec{q}_+ \approx \vec{f}(\vec{\rho}_{aq}, \vec{\pi}_{aq})$ as a consequence of the potentials appearing in the boosts. Therefore, the reconstruction of the relativistic orbit by means of equations (18) in terms of the relative motion is given by

$$\begin{aligned} &\vec{\eta}_i(\tau) \approx \vec{q}_+(\vec{\rho}_q, \vec{\pi}_q) + \frac{1}{2}\left[(-)^{i+1} - \frac{m_1^2 - m_2^2}{\mathcal{M}^2}\right]\vec{\rho}_q \to_{c\to\infty} \frac{1}{2}\left[(-)^{i+1} - \frac{m_1 - m_2}{m}\right]\vec{\rho}_q, \\ &\vec{\kappa}_i(\tau) \approx (-)^{i+1}\vec{\pi}_q(\tau), \\ &\qquad\qquad\Downarrow \\ &x_i^\mu(\tau) = z^\mu_{\mathrm{wigner}}(\tau, \vec{\eta}_i(\tau)) = u^\mu(P)\tau + \epsilon^\mu_r(u(P))\eta_i^r(\tau), \\ &p_i^\mu(\tau) = \sqrt{m_i^2 + \vec{\kappa}_i^2(\tau)}\,u^\mu(P) + \epsilon^\mu_r(P)\kappa_{ir}(\tau). \end{aligned} \tag{23}$$

While the potentials in $M_{(\mathrm{int})}$ determine $\vec{\rho}_q(\tau)$ and $\vec{\pi}_q(\tau)$ through the Hamilton equations, the potentials in $\vec{k}_{(\mathrm{int})}$ determine $\vec{q}_+(\vec{\rho}_q, \vec{\pi}_q)$. It is seen, therefore—as should be expected—that the relativistic theory of orbits is much more complicated than in the non-relativistic case, where the absolute orbits $\vec{\eta}_i(t)$ are proportional to the relative orbit $\vec{\rho}_q(t)$ in the rest frame.

3. A simple 2-particle model with a-a-a-d interaction

Instead of the physically more relevant but complicated system of [27], whose internal Hamiltonian and boosts in the rest-frame instant form are given in equation (10), let us study a simpler two-body system with an a-a-a-d interaction, defined at the end of appendix A in terms of two first class constraints. As we shall see, its treatment in the constraint formalism leads to a realization of the Poincare' algebra only in the rest frame. Therefore, let us look at its reformulation in the rest-frame instant form, where the rest-frame conditions are automatically contained.

In the rest-frame instant form, we may define the model by making the ansatz that the free generators of the internal realization of the Poincare' algebra given in equation (15) have the interaction forms

$$
\begin{aligned}
M_{(\mathrm{int})} &= \sqrt{m_1^2 + \vec{\kappa}_1^2 + \Phi(\vec{\rho}^2)} + \sqrt{m_2^2 + \vec{\kappa}_2^2 + \Phi(\vec{\rho}^2)},\\
\vec{p} &= \vec{\kappa}_1 + \vec{\kappa}_2,\\
\vec{j} &= \vec{\eta}_1 \times \vec{\kappa}_1 + \vec{\eta}_2 \times \vec{\kappa}_2,\\
\vec{k}_{(\mathrm{int})} &= -\vec{\eta}_1 \sqrt{m_1^2 + \vec{\kappa}_1^2 + \Phi(\vec{\rho}^2)} - \vec{\eta}_2 \sqrt{m_2^2 + \vec{\kappa}_2^2 + \Phi(\vec{\rho}^2)},
\end{aligned}
\tag{24}
$$

where $\vec{\rho} = \vec{\eta}_1 - \vec{\eta}_2$. Let us verify this ansatz by checking whether these generators satisfy the Poincare' algebra.

It is self-evident that one has

$$
[j_i, j_j] = \varepsilon_{ijk} j_k, \qquad [j_i, k_{(\mathrm{int})j}] = \varepsilon_{ijk} k_{(\mathrm{int})k}. \tag{25}
$$

We examine the other Poincare' brackets. First, note that

$$
\begin{aligned}
[p_i, k_{(\mathrm{int})j}] &= \Big[\kappa_{1i} + \kappa_{2i}, -\eta_{1j}\sqrt{m_1^2 + \vec{\kappa}_1^2 + \Phi(\vec{\rho}^2)} - \eta_{2j}\sqrt{m_2^2 + \vec{\kappa}_2^2 + \Phi(\vec{\rho}^2)}\Big]\\
&= \delta_{ij} M_{(\mathrm{int})}.
\end{aligned}
\tag{26}
$$

Next, examine $\left[\Phi'(x) = \frac{\mathrm{d}\Phi(x)}{dx}\right]$

$$
\begin{aligned}
[M_{(\mathrm{int})}, k_{(\mathrm{int})i}] &= \Big[\sqrt{m_1^2 + \vec{\kappa}_1^2 + \Phi(\vec{\rho}^2)} + \sqrt{m_2^2 + \vec{\kappa}_2^2 + \Phi(\vec{\rho}^2)},\\
&\quad - \eta_{1i}\sqrt{m_1^2 + \vec{\kappa}_1^2 + \Phi(\vec{\rho}^2)} - \eta_{2i}\sqrt{m_2^2 + \vec{\kappa}_2^2 + \Phi(\vec{\rho}^2)}\Big]\\
&= \kappa_{1i} + \kappa_{2i} - (\eta_{1i} + \eta_{2i})\Big[\sqrt{m_1^2 + \vec{\kappa}_1^2 + \Phi(\vec{\rho}^2)}, \sqrt{m_2^2 + \vec{\kappa}_2^2 + \Phi(\vec{\rho}^2)}\Big]\\
&= \kappa_{1i} + \kappa_{2i} - (\eta_{1i} + \eta_{2i})\left[-\frac{2\Phi'(\vec{\eta}^2)\vec{\rho}}{\sqrt{m_1^2 + \vec{\kappa}_1^2 + \Phi(\vec{\rho}^2)}} \cdot \frac{\vec{\kappa}_1 + \vec{\kappa}_2}{\sqrt{m_2^2 + \vec{\kappa}_2^2 + \Phi(\vec{\rho}^2)}}\right].
\end{aligned}
\tag{27}
$$

But the rest-frame condition $\vec{p} \approx 0$ implies

$$
[M_{(\mathrm{int})}, \vec{k}_{(\mathrm{int})}] \approx 0 \approx \vec{p}. \tag{28}
$$

The remaining crucial bracket is

$$
\begin{aligned}
[k_{(\mathrm{int})i}, k_{(\mathrm{int})j}] &= \Big[- \eta_{1i}\sqrt{m_1^2 + \vec{\kappa}_1^2 + \Phi(\vec{\rho}^2)} - \eta_{2i}\sqrt{m_2^2 + \vec{\kappa}_2^2 + \Phi(\vec{\rho}^2)},\\
&\quad - \eta_{1j}\sqrt{m_1^2 + \vec{\kappa}_1^2 + \Phi(\vec{\rho}^2)} - \eta_{2j}\sqrt{m_2^2 + \vec{\kappa}_2^2 + \Phi(\vec{\rho}^2)}\Big]\\
&= \eta_{1j}\kappa_{1i} - \eta_{1i}\kappa_{1j} + \eta_{2j}\kappa_{2i} - \eta_{2i}\kappa_{2j}\\
&\quad - \eta_{1i}\eta_{2j}\frac{2\Phi'(\vec{\rho}^2)\vec{\rho}\cdot(\vec{\kappa}_1 + \vec{\kappa}_2)}{\sqrt{m_1^2 + \vec{\kappa}_1^2 + \Phi(\vec{\rho}^2)}\sqrt{m_2^2 + \vec{\kappa}_2^2 + \Phi(\vec{\rho}^2)}}\\
&\quad + \eta_{2i}\eta_{1j}\frac{2\Phi'(\vec{\rho}^2)\vec{\rho}\cdot(\vec{\kappa}_1 + \vec{\kappa}_2)}{\sqrt{m_1^2 + \vec{\kappa}_1^2 + \Phi(\vec{\rho}^2)}\sqrt{m_2^2 + \vec{\kappa}_2^2 + \Phi(\vec{\rho}^2)}}.
\end{aligned}
\tag{29}
$$

Again, the rest-frame condition implies

$$
\begin{aligned}
[k_{(\mathrm{int})i}, k_{(\mathrm{int})j}] &\approx \eta_{1j}\kappa_{1i} - \eta_{1i}\kappa_{1j} + \eta_{2j}\kappa_{2i} - \eta_{2i}\kappa_{2j}\\
&= -\varepsilon_{ijk}\varepsilon_{klm}(\eta_{1l}\kappa_{1m} + \eta_{2l}\kappa_{2m}) = -\varepsilon_{ijk} j_k.
\end{aligned}
\tag{30}
$$

From equation (12), the interacting form of the canonical internal 3-center of mass is weakly equal to the 3-center of energy due to the rest-frame condition $\vec{p} \approx 0$

$$\vec{q}_+ \approx -\frac{\vec{k}_{(\text{int})}}{M_{(\text{int})}}. \tag{31}$$

Using the canonical transformations to relative variables given in equations (17), (18) (as well as equation (A.26) in appendix A) implies the following forms of $\vec{k}_{(\text{int})}$ and $M_{(\text{int})}$:

$$\begin{aligned}\vec{k}_{(\text{int})} &\approx -\vec{\eta}_1\sqrt{m_1^2+\vec{\pi}^2+\Phi(\vec{\rho}^2)} - \vec{\eta}_2\sqrt{m_2^2+\vec{\pi}^2+\Phi(\vec{\rho}^2)},\\ M_{(\text{int})} &\approx \sqrt{m_1^2+\vec{\pi}^2+\Phi(\vec{\rho}^2)} + \sqrt{m_2^2+\vec{\pi}^2+\Phi(\vec{\rho}^2)} = \mathcal{M}_{(\text{int})}.\end{aligned} \tag{32}$$

With $\mathcal{M} = \sqrt{m_1^2+\vec{\pi}^2} + \sqrt{m_2^2+\vec{\pi}^2}$, the canonical transformations in equations (18) in terms of free particle variables imply

$$\vec{\eta}_1 \approx \vec{q}_+ + \frac{1}{2}\left(1 - \frac{m_1^2-m_2^2}{\mathcal{M}^2}\right)\vec{\rho}, \qquad \vec{\eta}_2 \approx \vec{q}_+ - \frac{1}{2}\left(1 + \frac{m_1^2-m_2^2}{\mathcal{M}^2}\right)\vec{\rho}, \tag{33}$$

and also

$$\begin{aligned}\sqrt{m_1^2+\vec{\pi}^2} &= \frac{1}{2}(\mathcal{M}+\Delta) = \frac{\mathcal{M}}{2}\left(1 + \frac{m_1^2-m_2^2}{\mathcal{M}^2}\right),\\ \sqrt{m_2^2+\vec{\pi}^2} &= \frac{1}{2}(\mathcal{M}-\Delta) = \frac{\mathcal{M}}{2}\left(1 - \frac{m_1^2-m_2^2}{\mathcal{M}^2}\right),\end{aligned} \tag{34}$$

where $\Delta = \sqrt{m_1^2+\vec{\pi}^2} - \sqrt{m_2^2+\vec{\pi}^2}$, $\mathcal{M}\Delta = m_1^2 - m_2^2$. Therefore, we have the following expression for the 3-coordinates $\vec{\eta}_i$:

$$\begin{aligned}\vec{\eta}_1 &\approx \vec{q}_+ + \frac{1}{2}\left(1 - \frac{m_1^2-m_2^2}{\mathcal{M}^2}\right)\vec{\rho} = \vec{q}_+ + \frac{\sqrt{m_2^2+\vec{\pi}^2}}{\mathcal{M}}\vec{\rho},\\ \vec{\eta}_2 &\approx \vec{q}_+ - \frac{1}{2}\left(1 + \frac{m_1^2-m_2^2}{\mathcal{M}^2}\right)\vec{\rho} = \vec{q}_+ - \frac{\sqrt{m_1^2+\vec{\pi}^2}}{\mathcal{M}}\vec{\rho}.\end{aligned} \tag{35}$$

If we use the canonical basis $\vec{q}_+$, $\vec{p} \approx 0$, $\vec{\rho}$, $\vec{\pi}$ of the free case defined in equation (17) also in our simple interacting case, equations (35) must be replaced by equations (23). To this end we must find the functions $\vec{q}_+(\vec{\rho}, \vec{\pi})$ from the gauge conditions $\vec{q}_+^{(\text{int})} \approx 0$, namely from $\vec{k}_{(\text{int})} \approx 0$.

In our simple interacting case, substituting (35) into (32) and using equation (31) we get

$$\begin{aligned}-\mathcal{M}_{(\text{int})}\vec{q}_+^{(\text{int})} &\approx \vec{k}_{(\text{int})}\\ &\approx -\vec{q}_+\left[\sqrt{m_1^2+\vec{\pi}^2+\Phi(\vec{\rho}^2)} + \sqrt{m_2^2+\vec{\pi}^2+\Phi(\vec{\rho}^2)}\right]\\ &\quad - \vec{\rho}\,\frac{\sqrt{m_2^2+\vec{\pi}^2}\sqrt{m_1^2+\vec{\pi}^2+\Phi(\vec{\rho}^2)} - \sqrt{m_1^2+\vec{\pi}^2}\sqrt{m_2^2+\vec{\pi}^2+\Phi(\vec{\rho}^2)}}{\mathcal{M}}.\end{aligned} \tag{36}$$

If, in analogy to the free case, we define $\Delta_{(\text{int})} = \sqrt{m_1^2+\vec{\pi}^2+\Phi(\vec{\rho}^2)} - \sqrt{m_2^2+\vec{\pi}^2+\Phi(\vec{\rho}^2)}$, we have $\mathcal{M}_{(\text{int})}\Delta_{(\text{int})} = m_1^2 - m_2^2$ and we get

$$\sqrt{m_1^2+\vec{\pi}^{\,2}+\Phi(\vec{\rho}^2)} = \frac{1}{2}(\mathcal{M}_{(\mathrm{int})}+\Delta_{(\mathrm{int})}) = \frac{\mathcal{M}_{(\mathrm{int})}}{2}\left(1+\frac{m_1^2-m_2^2}{\mathcal{M}^2_{(\mathrm{int})}}\right),$$

$$\sqrt{m_2^2+\vec{\pi}^{\,2}+\Phi(\vec{\rho}^2)} = \frac{1}{2}(\mathcal{M}_{(\mathrm{int})}-\Delta_{(\mathrm{int})}) = \frac{\mathcal{M}_{(\mathrm{int})}}{2}\left(1-\frac{m_1^2-m_2^2}{\mathcal{M}^2_{(\mathrm{int})}}\right).$$

$$\Downarrow$$

$$\begin{aligned}\mathcal{M} = \sum_{i=1}^{2}\sqrt{m_i^2+\vec{\pi}^{\,2}} &= \sqrt{\left[\frac{\mathcal{M}_{(\mathrm{int})}}{2}\left(1+\frac{m_1^2-m_2^2}{\mathcal{M}^2_{(\mathrm{int})}}\right)\right]^2-\Phi(\vec{\rho}^2)}\\ &\quad+\sqrt{\left[\frac{\mathcal{M}_{(\mathrm{int})}}{2}\left(1-\frac{m_1^2-m_2^2}{\mathcal{M}^2_{(\mathrm{int})}}\right)\right]^2-\Phi(\vec{\rho}^2)}.\end{aligned} \tag{37}$$

As a consequence, equation (36) may be written in the form

$$\begin{aligned}\vec{k}_{(\mathrm{int})} &\approx -\mathcal{M}_{(\mathrm{int})}\left(\vec{q}_+ + \vec{\rho}\,\frac{\left(\sqrt{m_2^2+\vec{\pi}^{\,2}}-\sqrt{m_1^2+\vec{\pi}^{\,2}}\right)+\left(\sqrt{m_2^2+\vec{\pi}^{\,2}}+\sqrt{m_1^2+\vec{\pi}^{\,2}}\right)\frac{m_1^2-m_2^2}{\mathcal{M}^2_{(\mathrm{int})}}}{2\mathcal{M}}\right)\\ &= -\mathcal{M}_{(\mathrm{int})}\vec{q}_+ + \frac{m_1^2-m_2^2}{2}\left(\frac{\mathcal{M}_{(\mathrm{int})}}{\mathcal{M}^2}-\frac{1}{\mathcal{M}_{(\mathrm{int})}}\right)\vec{\rho}.\end{aligned} \tag{38}$$

Therefore the gauge fixing condition $\vec{q}_+^{(\mathrm{int})}\approx 0$, or $\vec{k}_{(\mathrm{int})}\approx 0$, gives

$$\vec{q}_+ \approx \vec{q}_+(\vec{\rho},\vec{\pi}) = \frac{m_1^2-m_2^2}{2}\left(\frac{1}{\mathcal{M}^2}-\frac{1}{\mathcal{M}^2_{(\mathrm{int})}}\right)\vec{\rho}, \tag{39}$$

so that, by using the inverse canonical transformation (18), in our simple interacting case we get the following reconstruction of the 3-coordinates $\vec{\eta}_i$ and of the 4-coordinates x_i^μ:

$$\begin{aligned}\vec{\eta}_1 &\approx \frac{1}{2}\left(1-\frac{m_1^2-m_2^2}{\mathcal{M}^2_{(\mathrm{int})}}\right)\vec{\rho} \to_{c\to\infty} \frac{m_2}{m_1+m_2}\vec{\rho},\\ \vec{\eta}_2 &\approx -\frac{1}{2}\left(1+\frac{m_1^2-m_2^2}{\mathcal{M}^2_{(\mathrm{int})}}\right)\vec{\rho} \to_{c\to\infty} -\frac{m_1}{m_1+m_2}\vec{\rho},\\ x_i^\mu(\tau) &= u^\mu(P)\tau + \frac{1}{2}\left[(-1)^{i+1}-\frac{m_1^2-m_2^2}{\sum_{j=1}^{2}\sqrt{m_j^2+\vec{\pi}^{\,2}+\Phi(\vec{\rho}^2)}}\right]\epsilon_r^\mu(P)\rho^r.\end{aligned} \tag{40}$$

This completes the reconstruction of the relativistic orbits of the two particles.

We see that in the non-relativistic limit, we recover the standard result in the center of mass frame $\vec{p}=0$. This is equivalent to adding the first class constraints $\vec{p}\approx 0$ to the non-relativistic Hamiltonian H_{com} and to fix the gauge by putting the non-relativistic center of mass into the origin

$$\vec{x} = \frac{m_1\vec{\eta}_1+m_2\vec{\eta}_2}{m_1+m_2}\approx 0. \tag{41}$$

For our simple interacting relativistic model, we get that the rest-frame 3-coordinates $\vec{\eta}_i$ are still proportional to the relative variable $\vec{\rho}$ (for more complex models, there could be a component along $\vec{\pi}$ coming from the function $\vec{q}_+(\vec{\rho},\vec{\pi})$). However, instead of the numerical proportionality constants of the non-relativistic case, we have a non-trivial dependence on the total constant fixed c.m. energy $\left(\mathcal{M}_{(\mathrm{int})}=\sqrt{m_1^2+\vec{\pi}^{\,2}+\Phi(\vec{\rho}^2)}+\sqrt{m_2^2+\vec{\pi}^{\,2}+\Phi(\vec{\rho}^2)}\right)$.

4. Evaluation of the orbits in the simple relativistic two-body problem with a Coulomb-like potential

For illustrative purposes, we make the following choice for the a-a-a-d potential Φ:

$$\Phi(\vec{\rho}^2) = -2\mu\frac{e^2}{\rho}, \qquad \rho = \sqrt{\vec{\rho}^2}. \tag{42}$$

This Coulomb-like potential is not to be confused with the real Coulomb potential between charged particles, which is outside the square roots as shown in equation (10) and which produces completely different relativistic orbits. However, both models may have the same non-relativistic limit for suitable choices of the parameters.

The invariant mass $M_{(\mathrm{int})}$ of the two-body model (the Hamiltonian of its relative motion) in the rest-frame instant form is

$$M_{(\mathrm{int})} \approx \mathcal{M}_{(\mathrm{int})} = \sqrt{m_1^2 + \vec{\pi}^2 - 2\mu\frac{e^2}{\rho}} + \sqrt{m_2^2 + \vec{\pi}^2 - 2\mu\frac{e^2}{\rho}}. \tag{43}$$

Instead of studying the Hamilton equations for $\vec{\rho}, \vec{\pi}$ with $\mathcal{M}_{(\mathrm{int})}$ as the Hamiltonian, we will find the orbits using Hamilton–Jacobi methods[4]. Since the potential is a central one, our orbit is confined to a plane with

$$\vec{\pi}^2 = \pi_\rho^2 + \frac{\pi_\phi^2}{\rho^2}. \tag{44}$$

Since both the time and the angle are cyclic, the generating function is

$$S = W_1(\rho) + \alpha_\phi\phi - wt \equiv W_1(\rho,\phi) - wt, \tag{45}$$

with w the invariant total c.m. energy. The Hamilton–Jacobi equation is

$$\sqrt{\left(\frac{\partial W_1}{\partial\rho}\right)^2 + \frac{\alpha_\phi^2}{\rho^2} + m_1^2 - 2\mu\frac{e^2}{\rho}} + \sqrt{\left(\frac{\partial W_1}{\partial\rho}\right)^2 + \frac{\alpha_\phi^2}{\rho^2} + m_2^2 - 2\mu\frac{e^2}{\rho}} = w. \tag{46}$$

This leads to (the function $b^2(w)$ is defined in equation (A.19) of appendix A)

$$\left(\frac{\partial W_1}{\partial\rho}\right)^2 + \frac{\alpha_\phi^2}{\rho^2} - 2\mu\frac{e^2}{\rho} = b^2(w), \tag{47}$$

and so we get

$$W_1(\rho,\phi) = \int \mathrm{d}\rho\sqrt{b^2(w) - \frac{\alpha_\phi^2}{\rho^2} + 2\mu\frac{e^2}{\rho}} + \alpha_\phi\phi. \tag{48}$$

The new coordinate canonically conjugate to the new momentum α_ϕ is the constant

$$\beta_2 = \frac{\partial W}{\partial\alpha_\phi} = -\int \frac{\alpha_\phi\,\mathrm{d}\rho}{\rho^2\sqrt{b^2(w) - \frac{\alpha_\phi^2}{\rho^2} + 2\mu\frac{e^2}{\rho}}} + \phi. \tag{49}$$

If we define

$$u = \frac{1}{\rho}, \tag{50}$$

[4] See [37] for the relativistic Kepler or Coulomb problem with respect to a fixed center and [38] for its use. Let us remark that the techniques of [37] could be applied to the results of [27] to describe the relativistic Darwin two-body problem. However, the internal boost $\vec{k}$ satisfying $\{p_i, k_j\} = \delta_{ij}M$ in the case of the pure Coulomb interaction $M = \sqrt{m_1^2 + \vec{\kappa}^2} + \sqrt{m_2^2 + \vec{\kappa}^2} + \frac{Q_1 Q_2}{4\pi|\vec{\eta}_1 - \vec{\eta}_2|}$ is not known.

we get

$$\beta_2 = \frac{\partial W}{\partial \alpha_\phi} = \int \frac{\alpha_\phi \mathrm{d}u}{\sqrt{b^2(w) - \alpha_\phi^2 u^2 + 2\mu e^2 u}} + \phi \tag{51}$$

or

$$\phi = \beta_2 = \int \frac{\mathrm{d}u}{\sqrt{\frac{b^2(w)}{\alpha_\phi^2} + \frac{2\mu e^2 u}{\alpha_\phi^2} - u^2}}. \tag{52}$$

This result leads to the ellipse (we consider only bounded orbits)

$$\frac{1}{\rho} = \frac{\mu e^2}{\alpha_\phi^2}\left[1 + \sqrt{1 + \frac{b^2(w)\alpha_\phi^2}{\mu^2 e^2}} \cos(\phi - \beta_2)\right]. \tag{53}$$

Equations (52) and (53) allow us to determine the orbit of the relative motion

$$\vec{\rho} = \rho(\cos\phi \mathbf{i} + \sin\phi \mathbf{j}). \tag{54}$$

Let us compare these results with the non-relativistic limit. In the non-relativistic Kepler or Coulomb case [1], equation (53) is replaced by the following expression (E is the non-relativistic energy):

$$\frac{1}{\rho} = \frac{\mu e^2}{\alpha_\phi^2}\left[1 + \sqrt{1 + \frac{2E\alpha_\phi^2}{\mu e^2}} \cos(\phi - \beta_2)\right]. \tag{55}$$

If we use the non-relativistic limit into equations (40) for the relation among $\vec{\eta}_i$ and $\vec{\rho}$, we get the non-relativistic expressions

$$\begin{aligned}
\vec{\eta}_1 &= \frac{\alpha_\phi^2}{m_1 e^2} \frac{1}{\left[1 + \sqrt{1 + \frac{2E\alpha_\phi^2}{\mu e^2}} \cos(\phi - \beta_2)\right]} (\cos\phi \mathbf{i} + \sin\phi \mathbf{j}), \\
\vec{\eta}_2 &= -\frac{\alpha_\phi^2}{m_2 e^2} \frac{1}{\left[1 + \sqrt{1 + \frac{2E\alpha_\phi^2}{\mu e^2}} \cos(\phi - \beta_2)\right]} (\cos\phi \mathbf{i} + \sin\phi \mathbf{j}).
\end{aligned} \tag{56}$$

For the relativistic counterparts, given in equation (40), we have from equations (37)

$$\begin{aligned}
w &= \mathcal{M}_{(\mathrm{int})} = \sum_{i=1}^{2} \sqrt{m_i^2 + \vec{\pi}^2 - 2\mu \frac{e^2}{\rho}}, \\
\mathcal{M} &= \sqrt{\left[\frac{w}{2}\left(1 + \frac{m_1^2 - m_2^2}{w^2}\right)\right]^2 + 2\mu \frac{e^2}{\rho}} \\
&\quad + \sqrt{\left[\frac{w}{2}\left(1 - \frac{m_1^2 - m_2^2}{w^2}\right)\right]^2 + 2\mu \frac{e^2}{\rho}} \equiv \mathcal{M}(w, \rho).
\end{aligned} \tag{57}$$

In this case, from equations (40) we have

$$\begin{aligned}
\vec{\eta}_1 &\approx \frac{1}{2}\rho(\cos\phi \mathbf{i} + \sin\phi \mathbf{j}) \left(1 - \frac{m_1^2 - m_2^2}{w^2}\right), \\
\vec{\eta}_2 &\approx -\frac{1}{2}\rho(\cos\phi \mathbf{i} + \sin\phi \mathbf{j}) \left(1 + \frac{m_1^2 - m_2^2}{w^2}\right),
\end{aligned} \tag{58}$$

where for ρ we have to use the solution given in equation (53).

We have the following situations.

(1) For equal masses, the relativistic and non-relativistic expressions are identical with $b^2(\mathcal{M}_{(\text{int})}) \mapsto 2\mu E$.

(2) In the limit in which one of the masses becomes very great (say m_2) then, since we have

$$\frac{m_1^2 - m_2^2}{w^2} \rightarrow_{m_2 \rightarrow \infty} -1, \tag{59}$$

the relativistic and non-relativistic expressions are also identical.

(3) If we introduce the new notation

$$w = \frac{m_1 + m_2}{\sqrt{\omega}}, \tag{60}$$

then the relativistic orbits become

$$\begin{aligned} \vec{\eta}_1 &= \frac{1}{2}\rho(\phi)(\cos\phi\mathbf{i} + \sin\phi\mathbf{j})\left(1 - \frac{m_1 - m_2}{m_1 + m_2}\omega\right), \\ \vec{\eta}_2 &= -\frac{1}{2}\rho(\phi)(\cos\phi\mathbf{i} + \sin\phi\mathbf{j})\left(1 + \frac{m_1 - m_2}{m_1 + m_2}\omega\right). \end{aligned} \tag{61}$$

Since ω is a constant of motion, the main difference between the relativistic and the non-relativistic orbits is the proportionality constant between individual particle coordinates and the relative coordinate, which, however, is now dependent on the invariant mass of the system.

5. Conclusions

We have given a complete treatment of the Hamiltonian two-body problem in the rest-frame instant form, arising from parametrized Minkowski theories when the dynamics is described with respect to the inertial intrinsic rest frame of the isolated system with its simultaneity 3-surfaces given by the Wigner hyper-planes. The existence of two realizations of the Poincare' group (the external one and the unfaithful internal one inside the Wigner hyper-planes), together with the clarification of the only three intrinsic notions of a center-of-mass-like collective variable, allow us to solve all the kinematical problems and to define canonical transformations for the separation of the center of mass from the relative motion as is possible in Newtonian mechanics.

In the rest-frame instant form of dynamics there is a natural gauge fixing $\vec{k}_{(\text{int})} \approx 0$ to the rest-frame conditions $\vec{p} \approx 0$, which allows us to completely clarify the determination of the relativistic orbits inside the Wigner hyper-planes. With this gauge fixing it is possible to describe the isolated system from the point of view of an inertial observer, whose worldline $x_s^\mu(\tau) = u^\mu(P)\tau$ is the (covariant non-canonical) Fokker–Pryce center of inertia. The simplest model with the a-a-a-d interaction is studied in detail.

To reconstruct the actual trajectories in Minkowski spacetime in the above inertial frame, we have to use equations (4)

$$\begin{aligned} x_i^\mu(\tau) &= u^\mu(P)\tau + \epsilon_r^\mu(P)\eta_i^r(\tau), \\ p_i^\mu(\tau) &= \sqrt{m_i^2 + \vec{\kappa}_i^2(\tau)}u^\mu(P) + \epsilon_r^\mu(P)\kappa_{ir}(\tau). \end{aligned} \tag{62}$$

To eliminate the momenta and to get a purely configurational description one should invert the first half of the Hamilton equations, $\dot{\vec{\rho}} = \{\vec{\rho}, \mathcal{M}_{(\text{int})}\}$, to get $\vec{\pi}$ in terms of $\vec{\rho}$ and $\dot{\vec{\rho}}$ $\left(\dot{f} = \frac{\mathrm{d}f}{\mathrm{d}\tau}\right)$.

Under Lorentz transformations Λ generated by the external Poincare' group, under which we have $\epsilon^\mu_r(u(\Lambda P)) = (R^{-1}(\Lambda, P))_r{}^s \Lambda^\mu{}_\nu \epsilon^\nu_s(u(P))$ and $\eta'^r_i = R^r{}_s(\Lambda, P)\eta^s_i$, the derived quantities x^μ_i and p^μ_i transform covariantly as 4-vectors[5]. However the worldlines $x^\mu_i(\tau)$ are not canonical variables, because they depend on the (noncanonical) Fokker–Pryce center of inertia. This, together with the non-covariance of the canonical center of mass $\tilde{x}^\mu$, is the way out from the no-interaction theorem in the rest-frame instant form.

Having understood both the kinematical and dynamical problems of relativistic orbit theory, the next step is to try to define a perturbation theory around relativistic orbits as has been done in the non-relativistic case [1]: it could be relevant for the special relativistic approximation of relativistic binaries in general relativity, till now treated only in the post-Newtonian approximation [39].

Appendix A. Two-body relativistic Hamiltonian mechanics with two first-class constraints

In constraint dynamics for classical relativistic spinless particles, one begins by introducing compatible generalized mass-shell constraints. We work with constraints that involve potentials that are independent of the relative momenta $\left(P^\mu = p^\mu_1 + p^\mu_2, M = \sqrt{P^2}, r^\mu = x^\mu_1 - x^\mu_2, \left\{x^\mu_i, p^\nu_j\right\} = -\delta_{ij}\eta^{\mu\nu}\right)$

$$\mathcal{H}_1 = p_1^2 - m_1^2 - \Phi_1(r, P) \approx 0, \qquad \mathcal{H}_2 = p_2^2 - m_2^2 - \Phi_2(r, P) \approx 0. \quad \text{(A.1)}$$

We call the scalars $\Phi_i(r, P)$ quasi-potentials (energy-dependent potentials that describe deviations from the free mass-shell conditions $p_i^2 - m_i^2 \approx 0$). Assuming that the model derives from an unknown reparametrization invariant Lagrangian (so that the canonical Hamiltonian vanishes), the Hamiltonian is defined in terms of the constraints only by

$$\mathcal{H} = \lambda_1(\tau)\mathcal{H}_1 + \lambda_2(\tau)\mathcal{H}_2, \quad \text{(A.2)}$$

in which λ_i are arbitrary Lagrange multipliers (called Dirac multipliers). The constraints forming this Hamiltonian must be such that the time rate of change of the constraints vanishes when the constraint is imposed. With the time rate of change of an arbitrary dynamical variable f given by

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = \{f, \mathcal{H}\}, \quad \text{(A.3)}$$

we get[6]

$$\frac{\mathrm{d}\mathcal{H}_1}{\mathrm{d}\tau} = \{\mathcal{H}_1, \mathcal{H}\} = \lambda_1(\tau)\{\mathcal{H}_1, \mathcal{H}_1\} + \lambda_2(\tau)\{\mathcal{H}_1, \mathcal{H}_2\} \approx \lambda_2(\tau)\{\mathcal{H}_1, \mathcal{H}_2\}, \quad \text{(A.4)}$$

and similarly

$$\frac{\mathrm{d}\mathcal{H}_2}{\mathrm{d}\tau} \approx \lambda_1(\tau)\{\mathcal{H}_2, \mathcal{H}_1\}. \quad \text{(A.5)}$$

[5] Namely the rest-frame instant form satisfies the *worldline condition*, since its synchronization of clocks (the one-to-one correlation between the worldlines) is a generalization of the gauge fixing $P \cdot (x_1 - x_2) \approx 0$ in models with second class constraints, as shown in [14]. As a consequence, the worldlines have an objective existence. However, in parametrized Minkowski theories one could choose different 3+1 splittings of Minkowski spacetimes corresponding to different one-to-one correlations (different conventions for the synchronization of clocks). Since each 3+1 splitting is equivalent to a different choice of the non-inertial frame [21] with its inertial forces (see appendix B of [3] for the non-inertial rest frames), the new worldlines will be different (they are obtained from those in the rest-frame instant form by means of a gauge transformation sending an inertial frame into a non-inertial one [22, 23]). This is the interpretation of the so-called *frame dependence of the worldlines* quoted in [14], where it was connected to a semantic problem.

[6] The Dirac multipliers, being functions only of τ, have zero Poisson bracket with phase space functions.

Thus, the constraints are constants (their time derivative weakly vanishes) provided that

$$\{\mathcal{H}_1, \mathcal{H}_2\} \approx 0. \tag{A.6}$$

This is called the compatibility condition. Thus, we must have

$$2p_1\{p_1, \Phi_2\} + 2p_2\{\Phi_1, p_2\} + \{\Phi_1, \Phi_2\} \approx 0. \tag{A.7}$$

We assume that the scalar functions Φ_i depend on the following variables:

$$\Phi_i = \Phi_i\left(\frac{r_\perp^2}{2}, \frac{r_{||}^2}{2}, M\right), \tag{A.8}$$

where

$$r_{||}^\mu = \frac{r\cdot P}{w^2}P^\mu, \qquad r_\perp^\mu = r^\mu - r_{||}^\mu, \qquad P\cdot r_\perp = 0. \tag{A.9}$$

Thus, our compatibility condition becomes

$$-4p_1\cdot r_\perp\frac{\partial\Phi_2}{\partial r_\perp^2} - 4p_1\cdot r_{||}\frac{\partial\Phi_2}{\partial r_{||}^2} - 4p_2\cdot r_\perp\frac{\partial\Phi_1}{\partial r_\perp^2} - 4p_2\cdot r_{||}\frac{\partial\Phi_1}{\partial r_{||}^2} + \{\Phi_1, \Phi_2\} \approx 0. \tag{A.10}$$

The simplest solution is

$$\Phi_1 = \Phi_2 = \Phi\left(\frac{r_\perp^2}{2}, M\right), \tag{A.11}$$

because it implies the following strong satisfaction of the compatibility condition:

$$\{\mathcal{H}_1, \mathcal{H}_2\} = -4P\cdot r_\perp\frac{\partial\Phi\left(\frac{r_\perp^2}{2}, M\right)}{\partial r_\perp^2} = 0. \tag{A.12}$$

This is the original Droz-Vincent, Todorov, Komar model [16–18]. More general forms of the functions Φ_i are possible for which $\{\mathcal{H}_1, \mathcal{H}_2\} \approx 0$, being proportional to the constraints themselves.

We define the canonical relative momentum by

$$q^\mu = \frac{\varepsilon_2 p_1^\mu - \varepsilon_1 p_2^\mu}{M}, \tag{A.13}$$

with

$$\varepsilon_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}, \qquad \varepsilon_2 = \frac{M^2 + m_2^2 - m_1^2}{2M}. \tag{A.14}$$

These constituent particle rest energies are defined so that

$$\varepsilon_1 + \varepsilon_2 = M, \qquad \varepsilon_1 - \varepsilon_2 = \frac{m_1^2 - m_2^2}{M}. \tag{A.15}$$

This definition is reinforced by

$$-\frac{p_1\cdot P}{M} = \frac{-P^2 + p_2^2 - p_1^2}{2M} \approx \varepsilon_1, \qquad -\frac{p_2\cdot P}{M} = \frac{-P^2 + p_1^2 - p_2^2}{2M} \approx \varepsilon_2, \tag{A.16}$$

(see equation (A.18)). Using $P^\mu = p_1^\mu + p_2^\mu$ and equation (A.13) gives

$$p_1^\mu = \frac{\varepsilon_1 P^\mu}{M} + q^\mu, \qquad p_2^\mu = \frac{\varepsilon_1 P^\mu}{M} - q^\mu. \tag{A.17}$$

In term of these variables, the difference of the constraints depends on the relative energy in the rest frame

$$\mathcal{H}_1 - \mathcal{H}_2 = p_1^2 + m_1^2 - p_2^2 - m_2^2 = 2P\cdot q \approx 0, \tag{A.18}$$

where we have used

$$\varepsilon_1^2 - m_1^2 = \varepsilon_2^2 - m_2^2 = \frac{1}{4M^2}\big(M^4 - 2\big(m_1^2 + m_2^2\big)M^2 + (m_1^2 - m_2^2)^2\big) \equiv b^2(M). \qquad \text{(A.19)}$$

On the other hand, the sum of the two constraints determines the mass spectrum of the two-body system. It can be written as

$$q^2 + \Phi\left(\frac{r_\perp^2}{2}, M\right) - b^2(M) \approx 0 \qquad \text{(A.20)}$$

or

$$\vec{q}^{\,2} + \Phi\left(\frac{\vec{r}^{\,2}}{2}, M\right) - b^2(M) \approx 0, \qquad \text{(A.21)}$$

in the rest frame (where $q^o \approx 0$ and $r_\perp^2 \approx \vec{r}^{\,2}$). To get the mass spectrum, this equation has to be solved for $M = \sqrt{P^2}$.

Since we have $\{x_i^\mu, \mathcal{H}_1\} \neq 0, \{x_i^\mu, \mathcal{H}_2\} \neq 0$, Droz-Vincent covariant non-canonical positions [14, 16] q_i^μ are defined as the solutions of the two equations $\{q_1^\mu, \mathcal{H}_2\} = 0$ and $\{q_2^\mu, \mathcal{H}_1\} = 0$.

If in equation (A.21) we consider a M-independent potential in the rest frame, we get that the free expression

$$\vec{q}^{\,2} = \frac{1}{4M^2}\big[M^4 - 2\big(m_1^2 + m_2^2\big)M^2 + \big(m_1^2 - m_2^2\big)^2\big] \qquad \text{(A.22)}$$

is modified to

$$\vec{q}^{\,2} + \Phi\left(\frac{1}{2}\vec{r}^{\,2}\right) = \frac{1}{4M^2}\big[M^4 - 2\big(m_1^2 + m_2^2\big)M^2 + \big(m_1^2 - m_2^2\big)^2\big], \qquad \text{(A.23)}$$

which is the rest-frame form of a covariant two-body constraint dynamics [24] involving two generalized mass-shell constraints of the form

$$p_1^2 - m_1^2 - \Phi \approx 0, \qquad p_2^2 - m_2^2 - \Phi \approx 0, \qquad \text{(A.24)}$$

with

$$\Phi = 2\mu V\left(\frac{1}{2}\vec{r}^{\,2}\right), \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}, \qquad \text{(A.25)}$$

i.e. a form suitable for the non-relativistic limit.

Solving equation (A.23) algebraically for M and choosing all positive square roots leads to[7]

$$M = \sqrt{m_1^2 + \vec{q}^{\,2} + \Phi\left(\frac{1}{2}\vec{r}^{\,2}\right)} + \sqrt{m_2^2 + \vec{q}^{\,2} + \Phi\left(\frac{1}{2}\vec{r}^{\,2}\right)}. \qquad \text{(A.26)}$$

In section 3, the rest-frame instant form of this model is studied in detail. In particular, the form of the generators of the internal Poincare' group is given.

[7] The choice of equation (A.13) for the relative momentum is the relativistic generalization of $\vec{q} = (m_2\vec{p}_1 - m_1\vec{p}_2)/(m_1 + m_2) = \mu\,\mathrm{d}\vec{r}/\mathrm{d}t$. The alternative choice of $q^\mu = (p_1^\mu - p_2^\mu)/2$ would lead to the constraint $2P \cdot q = m_1^2 - m_2^2$ in place of equation (A.18). However it would lead to the same result, equation (A.23), for $\vec{q}^{\,2}$ even for unequal mass since the relative energy is not zero for this choice of q unlike for that given in equation (A.13). Hence, the expressions in equations (A.26) and (32) for the c.m. energy are the same with both choices of the relative momentum.

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